DA-IICT, Autumn 2015-16

CT203 Signals & Systems **First In-Sem examination**

Maximum marks 25 11 September 2015, Duration: 90 mins.

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1. The *absent* symbol has been ignored in the set of input and output symbols.
2. Unless otherwise stated, assume the usual definitions for vector space addition and scalar multiplication.

1. A water pump supplies water into a water tank based on the amount of water present in the tank. Two transducers, one at height h_1, the other at height h_2, ($h_1 > h_2$), each of which produces a '1' whenever the water level is greater than the respective heights, otherwise a '0'. The water pump should switch on when the water level goes below h_2 and should turn off only after the level crosses h_1. Similarly, once the pump switches off after the level crosses h_1, it should turn on only after the level goes below h_2. Design an FSM that will control the water pump as per the above requirements. If any FSM is unable to do the job, give a proof. Assume that the tank is initially empty. **[5 marks]**

2. Consider FSM's with input and output symbols given by $R = \{True, False\}$.
[2+2+3=7 marks]

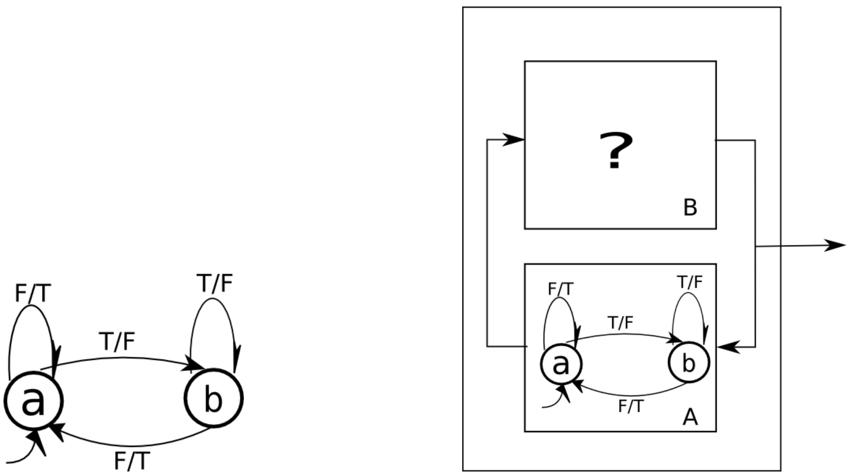

Figure 1: (left) FSM - system A, (right) Cascade and feedback composition of systems A and B.

   (a) For the FSM shown in Figure 1 (left), find the relation between the input and output sequence. Let the output of this system (henceforth called system A) be fedback into its input. Is the system well-formed?

   (b) Can system A be simulated by an FSM with smaller number of states? If yes, provide the State transition diagram of this FSM along with the simulation relation. In case it cannot be simulated by a simpler FSM, give precise reasons for the same.

(c) Consider the cascade and feedback composition of system A with an unknown system B, as shown in Figure 1(right). Draw the state transition diagram of system B, such that this composition is well-formed and the output of the system to a sequence of *React* inputs is a sequence of alternating *True* and *False* symbols, beginning with a *True* symbol. Provide precise reasons behind your choice of system B.

3. Let the set of input and output symbols be $R = \{0, 1\}$. Design an FSM whose output y at index n is 1 if the input sequence x till index n, $(x_n, x_{n-1}, \ldots, x_1, x_0)$ is a palindrome (a sequence is a palindrome if it is the same sequence read left to right or right to left), and a 0 if not. If it is not possible for an FSM to do so, then provide a proof. **[3 marks]**

4. Which of the following systems T are linear? For any T which is linear, represent T as a matrix corresponding to your choice of basis. **[2+2+2 = 6 marks]**

   (a) $T : \mathbb{R}^n \to \mathbb{R}^n$, where T outputs the input sequence in ascending order.

   (b) $T : \mathbb{C} \to \mathbb{C}$, from and to the vector space $\mathbb{C}$ over $\mathbb{R}$(set of scalars), defined as $T(x + jy) = x - jy, \forall x + jy \in \mathbb{C}$.

   (c) $T : \mathbb{C} \to \mathbb{C}$, from and to the vector space $\mathbb{C}$ over $\mathbb{C}$(set of scalars), defined as $T(x + jy) = x - jy, \forall x + jy \in \mathbb{C}$.

5. Let $R = \{0, 1\}$ represent the set of symbols for input and output sequences. Let h be a fixed sequence with $h(2) = 1, h(1) = 0, h(0) = 1$, with zero everywhere else. Design an FSM that has the following relation between the input sequence x and output sequence y:

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k), \forall n \geq 0,$$

where the summation $\sum$ is done modulo-2, for example, $\sum_{k=0}^{2} x(k)h(n-k) = x(0)h(n) +_2 x(1)h(n-1) +_2 x(2)h(n-2)$. You may assume that the input sequence is zero for $n < 0$ and does not terminate. **[4 marks]**

Solutions

1. The transducers produce a continuous function with values in $R = \{0, 1\}$. Let us denote the functions thus obtained from the transducers at height h_1 and h_2 by $x_1(t)$ and $x_2(t)$ for $t \geq 0$. These two functions are the input to the pump controller which will output a function denoted by $y(t), t \geq 0$.
Thus the set of input and output signals are

$$Inputs = \left[\mathbb{R}^+ \to \mathbb{Z}_2\right] \times \left[\mathbb{R}^+ \to \mathbb{Z}_2\right]$$
$$Outputs = \left[\mathbb{R}^+ \to \mathbb{Z}_2\right]$$

Now, when the inputs are of the form $(x_1(t), x_2(t)) = (0, 0)$, irrespective of anything else, the pump should be turned on, i.e., $y(t) = 1$, similarly for $(x_1(t), x_2(t)) = (1, 1)$, $y(t) = 0$. Only when inputs are of the form $(x_1(t), x_2(t)) = (0, 1)$ the output depends on whether the water level is going down or up. Since the water level in this case is between h_1 and h_2, either it must have started from height greater than h_1 or height smaller than h_2. This is what the state of the system should encode. Let $States = \{UP, DOWN\}$, where the state UP implies the pump has been switched on and the water level is rising, while $DOWN$ implies the pump has been switched off and the water level is falling. When the inputs are $(0, 1)$, in state UP, the output should be $y(t) = 1$, while if the state is $DOWN$, the output should be $y(t) = 0$. Since the tank is assumed to empty initially, $s(0) = UP$ (although the system can be in any state, the initial state here does not matter). The input $(1, 0)$ is never possible and can be thought of as a *don't care* condition. The obvious assumption is that the transducers are working as they should. The update table of this FSM is,

Current state	Input	Output	Next state
UP	00	1	UP
	01	1	UP
	10 (never occurs)	1	UP
	11	0	DOWN
DOWN	00	1	UP
	01	0	DOWN
	10 (never occurs)	0	DOWN
	11	0	DOWN

2. Cascade and feedback FSM:

   (a) The input-output relation is simply $y(n) = \overline{x(n)}$, the complement. As can be seen in the relation, the FSM with feedback will not be well-formed since it lacks any fixed point in any state.

   (b) From the input-output relation, it is clear that the system does not need anything other than the current input $x(n)$, hence the system can be simulated by an FSM with one state. The update table for this system is

Current state	Input	Output	Next state
a1	T	F	a1
a1	F	T	a1

   The simulation relation is given by $S = \{(a, a1), (b, a1)\}$.

   (c) Let the unknown system be denoted by B. We first find the cascade composition $C = B \circ A$ and then feedback output of system C to its own input.
   It is given that the feedback system's output sequence begins with a T. The system A begins in the initial state a. Let us say, the system B has an initial state c.

Then for the state *ac* of the cascade system, *T* has to be the unique fixed. This allows us to determine the output of the state *ac* for any inputs, i.e., for input *T*, the output of system *C* in state *ac* should be *T*, while for an input *F*, the output should be *T* (otherwise state *ac* will have two fixed points). For this to happen, system *B* in its state *c* should output *T* for an input *F*, and output *T* for an input *T*. Thus state *ac* will have a unique fixed point of *T* which is the desired first output of the feedback system. After producing a *T*, the state of system *A* becomes *b*. Now if there are no other states in *B*, then *T* is still a fixed point for system *C* (with feedback) when in state *bc*, thus sytem *B* will contain at least one more state, let's call it *d*. Since the system *B* in state *c* always produces a *T* (state determined output), in order to produce the next output of *F* for the feedback composed system *C*, the system *B* will move to state *d* when the input to system *B* in state *c* is *F*. Thus the system *C* is now in state *bd* when the input to system *A* is *T*. As given in the requirement, in this state the unique fixed point should be *F*, and once this fixed point is produced in the output, the system should move back to state *ac* (for producing the next *T*). For the fixed point to be *F*, the output of system *B* in state *d*, for input *T* should be *F* (since this is the output of system *A* in state *b* for input *F*) and *F* for input *T*. Moreover after it produces an output *F* for input *T*, the system *B* should move into state *c*, while it stays in state *d* for input *F*. Following is the update table for system *B* (which is a state-determined output system)

Current State	Input	Output	Next State
c	T	T	c
c	F	T	d
d	T	F	c
d	F	F	d

Following is the update table for the cascade system *C*:

Current State	Input	Output	Next State
ac	T	T	bd
	F	T	ac
ad	T	F	bd
	F	F	ac
bc	T	T	bd
	F	T	ac
bd	T	F	bd
	F	F	ac

As can be seen, the states *bc* and *ad* are not reachable, while the other two states *ac* and *bd* have a unique fixed point *T* and *F* respectively, and since the feedback system begins in state *ac*, it will produce an alternating sequence of *T* and *F* starting with *T*.

3. Let us assume that an FSM with N states will be able to detect whether the input sequence till any time n is a palindrome or not. Consider the input sequence 1^N. Since there are only N states in the FSM, for this input sequence, the system will be in the same state at two distinct instants of time, say p and q, $s(p) = s(q), p > q$. Once two copies of the system is in the same state, for the same input the output produced by both copies has to be the same. Therefore the output produced by the system for inputs $1^p0^p1^p$ and $1^p0^p1^q$ will be the same, which contradicts the fact that the system detects palindromes.

4. Linearity:

(a) $T : \mathbb{R}^n \to \mathbb{R}^n$: The system is a sorting algorithm. It is obvious to see that $T(\delta_1) = \delta_n$ and $T(-1 \cdot \delta_1) = -\delta_1 \neq -1 \cdot \delta_n$, hence T does not satisfy homogeneity and is therefore not linear.

(b) $T(x + jy) = x - jy, \forall x + jy \in \mathbb{C}$, where $\mathbb{C}$ is a vector space over $\mathbb{R}$.
Additivity:

$$\begin{aligned} T((a+jb)+(c+jd)) =& T((a+c)+j(b+d)) \\ =& (a+c) - j(b+d) \\ =& (a-jb)+(c-jd) \\ =& T(a+jb)+T(c+jd), \forall (a+jb),(c+jd) \in \mathbb{C} \end{aligned}$$

Homogeneity:

$$\begin{aligned} T(k \cdot (a+jb)) =& T(ak+jbk) \\ =& ak - jbk \\ =& k(a-jb) \\ =& k \cdot T(a+jb), \forall k \in \mathbb{R}, \forall (a+jb) \in \mathbb{C} \end{aligned}$$

Thus T is linear. Let the basis of $\mathbb{C}$ over $\mathbb{R}$ be $B = \{1, j\}$. Since $T(1) = 1 = 1 + 0j$ and $T(j) = -j = 0 + j(-1)$, the matrix representation for T in B becomes

$$[T]_B^B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(c) The same T but the vector space is $\mathbb{C}$ over $\mathbb{C}$. Let us check homogeneity:

$$\begin{aligned} T((p+jq)\cdot(a+jb)) =& T((pa-qb)+j(qa+pb)) \\ =& (pa-qb) - j(qa+pb) \\ =& p(a-jb) + jq(-a+jb) \\ \neq& (p+jq)T(a+jb) \end{aligned}$$

Therefore T is not linear between the given vector spaces.

5. Let $y(n) = \sum_{k=0}^{n} x(k)h(n-k)$. We know convolution is commutative, hence $y(n) = \sum_{k=0}^{n} h(k)x(n-k)$. With the given sequence h, we get $y(n) = h_0 x(n) +_2 h_1 x(n-1) +_2 h_2 x(n-2)$. This simplifies to $y(n) = x(n) +_2 x(n-2)$.
In order to implement this relation as an FSM, the system will require to store $x(n-1)$ and $x(n-2)$, giving us the states of the system as $s(n) = (x(n-1), x(n-2))$. The set of input symbols and output symbols is R, while the states are $R \times R$. The initial state is $s(0) = (0,0)$, and the update table is

Current State	Input	Output	Next state
00	0	0	00
	1	1	10
01	0	1	00
	1	0	10
10	0	0	01
	1	1	11
11	0	1	01
	1	0	11