SC105- Calculus and Complex Variables

Home Work 3 Solutions

Week: August 24, 2015

Tutorial Discussion Week: August 24, 2015

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Q.1 Find the derivatives (if exist any) of the following functions:

(1) $f(x) = x|x|, x \in \mathbb{R}$

Solution: We have

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x \leq 0 \end{cases}$$

Now for an arbitrary point $c \in \mathbb{R}$, we have

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\ &= \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \\ &= \lim_{h\to 0} \frac{(c+h)^2-c^2)}{h} \quad \text{if } c > 0 \\ &= \lim_{h\to 0} \frac{(h)(h+2c)}{h} \\ &= 2c \end{aligned}$$

or

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{-(c+h)^2-(-c^2)}{h} \quad \text{if } c < 0 \\ &= \lim_{h\to 0} \frac{-h^2-2ch}{h} \\ &= -2c \end{aligned}$$

$$\Rightarrow f'(x) = \begin{cases} 2x & \text{if } x > 0 \\ -2x & \text{if } x < 0 \end{cases}$$

Interesting case will be for $x = 0$, as we know that $|x|$ is not differentiable at $x = 0$
By given definition,

$$f'_+(0) = \lim_{h\to 0^+} \frac{h^2-0}{h} = 0$$

$$f'_-(0) = \lim_{h\to 0^-} \frac{-h^2-0}{h} = 0$$

$\Rightarrow f(x)$ is differentiable at $x = 0$

$\Rightarrow f(x)$ is differentiable at $x \in \mathbb{R}$.

(2) $f(x) = \sqrt{|x|}, x \in \mathbb{R}$

Solution:

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

$$= \lim_{h \to 0} \frac{f(x+h)-f(c)}{h}$$

for $x > 0$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{|x+h|}-\sqrt{|x|}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \text{if} \quad x > 0$$

$$= -\frac{1}{2\sqrt{x}} \quad \text{if} \quad x < 0$$

for $x = 0$

$$f'_{+}(0) = \lim_{h \to 0^{+}} \frac{\sqrt{h}-0}{h} = +\infty$$

$$f'_{-}(0) = \lim_{h \to 0^{-}} \frac{\sqrt{-h}-0}{h} = -\infty$$

$$\Rightarrow \quad f'_{+}(0) \neq f'_{-}(0)$$

$\Rightarrow$ The derivative of $f(x)$ at $x = 0$ does not exist.

(3) $f(x) = \log_x 2, x > 0, x \neq 1$

Solution:

$$\begin{aligned} f(x) &= \log_x 2 \\ &= \frac{ln2}{\ln x} \\ f'(x) &= -\frac{\ln 2}{x \ln x^2} \\ &= -\frac{\log_x 2 \log_x e}{x} \end{aligned}$$

. . . .

Q2 Assume that f is continuous on $[a, b]$, $a > 0$ and differentiable on (a, b). Show that if

$$\frac{f(a)}{a} = \frac{f(b)}{b}$$

then there is $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$

Solution:

we know the roll's theorm. If the real valued function f is continuous on $[a, b]$ and differentiable on (a, b) then f attains a maximum or minimum value at any point $x_0 \in (a, b)$ i.e. $f'(x_0) = 0$.
Given

$$\frac{f(a)}{a} = \frac{f(b)}{b}$$

Let $h(x)$ be defined as: $h(x) = \frac{f(x)}{x}$
As $f(x)$ is continuous on $[a, b]$, $a > 0$, $h(x)$ will also continuous on $[a, b]$
(possibly if $x = 0$ this will not be true)
similarly $h(x)$ is also differentiable on (a, b)

$$\Rightarrow h(a) = \frac{f(a)}{a} = \frac{f(b)}{b} = h(b)$$

From Roll's theorm

$$\exists x_0 \text{ such that } h'(x_0) = 0$$

$$\begin{aligned} h(x) &= \frac{f(x)}{x} \\ \Rightarrow \quad h'(x) &= \frac{x f'(x) - f(x)}{x^2} \\ \Rightarrow \quad h'(x_0) &= \frac{x_0 f'(x_0) - f(x_0)}{x_0^2} = 0 \\ \Rightarrow \quad x_0 f'(x_0) - f(x_0) &= 0 \qquad (x \neq 0) \\ \Rightarrow \quad x_0 f'(x_0) &= f(x_0) \end{aligned}$$

. . . .

Q3 Suppose f is continuous in$[a, b]$ and differentiable in (a, b). Given

$$f^2(b) - f^2(a) = b^2 - a^2,$$

then the equation$f'(x)f(x) = x$ has at least one root in(a, b)

Solution:

Let $h(x) = f^2(x) - x^2, x \in [a, b]$

As f is continuous on $[a, b]$ and differentiable on (a, b), h is also continuous on $[a, b]$ and differentiable on(a, b)

$$h(a) = f^2(a) - a^2$$

$$h(b) = f^2(b) - b^2$$

$$\text{Given, } f^2(b) - b^2 = f^2(a) - a^2$$

$$\Rightarrow h(a) = h(b)$$

From roll's theorm,

$$\exists x_0 \in (a, b \text{ such that } h'(x_0) = 0$$

$$\begin{array}{rrcll} & h(x) & = & f^2(x) - x^2 & \\ & h'(x) & = & 2f(x)f'(x) - 2x & \\ \Rightarrow & h'(x_0) & = & 2f'(x_0)f(x_0) - 2x_0 & = \quad 0 \\ \Rightarrow & f'(x_0)f(x_0) & = & x_0 & \end{array}$$

. . . .

Q4 Assume that $a_0, a_1, a_2, \cdots, a_n$ are real number such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_n - 1}{2} + an = 0$$

prove that the polynomial

$$p(x) = a_0 x_n + a_1 x^n - 1 + \cdots + a_n$$

has at least one root in $(0, 1)$

Solution:

One can see that p(x) is the differentiation of a function

$$Q(x) = \frac{a_0}{n+1}x^{n+1} + \frac{a_1}{n}x^n + \cdots + \frac{a_n - 1}{2}x^2 + anx = 0$$

If we prove that $Q(x)$ is differntiable on $[0, 1]$ then from the Roll's theorm there must be $x_0 \in (0, 1)$ satisfying $q'(x) = 0$ i.e $p(x) = 0$

So our problem is to define $Q(x)$ and prove it to be differntiable on $[0, 1]$

Here

$$Q(0) = 0 = Q(1) \text{ (easy to see)}$$

$$\&Q'(x) = \lim_{h \to 0} \frac{Q(x+h) - Q(x)}{h} \text{ exist}$$

∴ Q(x) is differentiable & Roll's theorm can be applied.

. . . .

Q4 Find the Taylor series expansion of f about zero when

(a) $f(x) = \sin^6 x + \cos^6 x, x \in \mathbb{R}$

Solution: Taylor series expansion:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

Here,

$$f(x) = \sin^6 x + \cos^6 x$$
$$= \frac{5}{8} + \frac{3}{8}\cos 4x, x \in \mathbb{R}$$

and

$$\cos x = \sum_{n=0}^{\infty}(-1)^{2n}\frac{x^{2n}}{(2n)!}, x \in \mathbb{R}$$

consequently,

$$\sin^6 x + \cos^6 x = \frac{5}{8} + \frac{3}{8}\sum(-1)^n 4^{2n}\frac{x^{2n}}{(2n)!}, x \in \mathbb{R}$$

Note that one can see that the Taylor series of $\cos(x)$ is as given since all the required conditions for its existence holds true.

(b) $f(x) = \ln\frac{(1+x)}{(1-x)}, x \in (-1, 1)$

Solution:

The key idea is to break it into two series and then use the Taylor series for $\ln(1+x)$.

$$f(x) = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad x \in (-1, 1)$$
$$= \frac{1}{2}[\ln(1+x) - \ln(1-x)]$$

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \quad x \in (-1, 1)$$