SC116- Algebraic Structures

Home Work 8

Week: September 23, 2013

Tutorial Discussion Week: September 30, 2013

(1) If G is a group such that $(ab)^i = a^i b^i$ for three consecutive integers i for all $a, b \in G$, show that G must be an abelian group.

(2) Determine the butterfly (lattice diagram) for all the subgroups of S_3.

(3) Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}_p$, p a prime number such that $ad - bc \neq 0$. G forms a group with respect to matrix multiplication. What is $o(G)$? Let H be a subgroup of G defined by $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \mid ad - bc = 1 \right\}$. What is $o(H)$?

(4) Write out all the right cosets of H in G and all the left cosets of H in G where (a) $G =< a >$ is a cyclic group of order 10 and $H =< a^2 >$ is the subgroup of G generated by a^2. (b) $G =< a >$ is a cyclic group of order 10 and $H =< a^5 >$ is the subgroup of G generated by a^5. Is every right cosets of H in G a left cosets of H in G?

(5) If $a \in G$, define $N(a) = \{x \in G \mid xa = ax\}$. Show that $N(a)$ (known as Normalizer of a in G) is a subgroup of G.

(6) If H is a subgroup of G, show that $C(H) = \{x \in G \mid xh = hx \text{ for all } h \in H\}$ is a subgroup of G. (C(H) is known as Centralizer of H in G).

(7) The center $\mathbb{Z}$ of a group G is defined by $\mathbb{Z} = \{\alpha \in G \mid \alpha x = x\alpha \text{ for all } x \in G\}$. Show that $\mathbb{Z}$ is a subgroup of G.

(8) If H is a subgroup of G, let $N(H) = \{a \in G \mid aHa^{-1} = H\}$. Show that $N(H)$ is a subgroup of G and $H \subset N(H)$.

(9) For the group S_3, compute $N(a)$ for every $a \in S_3$, compute $C(H)$ and $N(H)$ for every subgroup H of S_3. Also compute the center $\mathbb{Z}$ of S_3.

(10) Give an example of a group G and a subgroup H such that $N(H) \neq C(H)$. Is there any containing relation between $N(H)$ and $C(H)$?

(11) Let U_n denote the integers relatively prime to n under multiplication $mod\ \ n$. Show that U_n is a group. Also show the following:

(a) U_8 is not a cyclic group. (b) U_9 is a cyclic group. What are all its generators? (c) Guess at what values of n, U_n is a cyclic group.