SC116- Algebraic Structures

Home Work 6

Week: September 2, 2013

Tutorial Discussion Week: September 9/16, 2013

(1) Let $<,>$ denote the standard inner product on $\mathbb{C}^n(\mathbb{C})$ and let $A \in M_n(\mathbb{C})$, ie, $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^\star \mathbf{y} \in \mathbb{C}^n$. Show that $< A\mathbf{x}, \mathbf{y} >=< \mathbf{x}, A^\star \mathbf{y} >$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

(2) Let A be an $n \times n$ orthogonal matrix. Show that

(a) the rows of A forms an orthonormal basis of $\mathbb{R}^n$. (b) the columns of A forms an orthonormal basis of $\mathbb{R}^n$. (c) For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n\times 1}$, $< A\mathbf{x}, A\mathbf{y} >=< \mathbf{x}, \mathbf{y} >$. (d) For any vector $\mathbf{x} \in \mathbb{R}^{n\times 1}$, $\|A\mathbf{x}\| = \|\mathbf{x}\|$.

(3) Let A be an $n \times n$ unitary matrix. Show that

(a) the rows/columns of A forms an orthonormal basis of $\mathbb{C}^n$. (b) For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{n\times 1}$, $< A\mathbf{x}, A\mathbf{y} >=< \mathbf{x}, \mathbf{y} >$. (c) For any vector $\mathbf{x} \in \mathbb{C}^{n\times 1}$, $\|A\mathbf{x}\| = \|\mathbf{x}\|$.

(4) Show that the product of two $n\times n$ orthogonal (unitary) matrices is orthogonal (unitary) matrix.

(5) Find the eigen-pairs for each of the following matrices.

$$\begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}, \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(6) Does the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, for some $\theta, 0 \le \theta \le 2\pi$ diagonalizable?

(7) Let $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ be a linear operator with $rank(T - I) = 3$ and $N(T) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 + x_4 + x_5 = 0, x_2 + x_3 = 0\}$. Determine the eigenvalues of T? Find the number of linearly independent eigenvectors corresponding to each eigenvalue? Is T diagonalizable? Justify your answer.

(8) Let A be a square matrix such that $UAU^\star$ is a diagonal matrix for some unitary matrix U. Show that A is a normal matrix.

(9) Let $A \in M_n(\mathbb{C})$. Show that $A - A^\star$ is always skew-Hermitian matrix.

(10) Show that the matrices $A = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$, and $B = \begin{bmatrix} 10 & 9 \\ -4 & -2 \end{bmatrix}$ are similar. Is it possible to find a unitary matrix U such that $A = U^\star BU$?