# SC116- Algebraic Structures

Home Work 5

**Week**: August 26, 2013

**Tutorial Discussion Week**: September 2, 2013

(1) For $\mathbf{x} = (x_1, x_2)$ & $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$, define the three maps as follows. Identify if they really define the IPS. If not, find out which condition out of 3 conditions for IPS they fails to satisfy. Give reasons for your answers.
(a) $< \mathbf{x}, \mathbf{y} >= x_1 y_1$. (b) $< \mathbf{x}, \mathbf{y} >= x_1^2 + y_1^2 + x_2^2 + y_2^2$. (c) $< \mathbf{x}, \mathbf{y} >= x_1 y_1^3 + x_2 y_3^2$.
(2) Show that $< \mathbf{x}, \mathbf{y} >= 10x_1y_1 + 3x_1y_2 + 3x_2y_1 + 2x_2y_2 + x_2y_3 + x_3y_2 + x_3y_3$. defines an inner product in $\mathbb{R}^3$, where $\mathbf{x}, \mathbf{y}, \in \mathbb{R}^3$.
(3) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, prove the following identities:
(a) $||\mathbf{x} + \mathbf{y}||^2 + ||\mathbf{x} - \mathbf{y}||^2 = 2(||\mathbf{x}||^2 + ||\mathbf{y}||^2)$. (b) $< \mathbf{x}, \mathbf{y} >= 0 \iff ||\mathbf{x} - \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$. (c) $||\mathbf{x}|| = ||\mathbf{y}|| \iff < \mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y} >= 0$. (d) $4 < \mathbf{x}, \mathbf{y} >= ||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2$. (e) $||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.
Are the above results true if $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n(\mathbb{C})$?
(4) Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n(\mathbb{C})$, prove the following identities:
(a) $4 < \mathbf{x}, \mathbf{y} >= ||\mathbf{x} + \mathbf{y}||^2 - ||\mathbf{x} - \mathbf{y}||^2 + i||\mathbf{x} + i\mathbf{y}||^2 - i||\mathbf{x} - i\mathbf{y}||^2$. (b) If $\mathbf{x} \neq 0$, then $||\mathbf{x} + i\mathbf{x}||^2 = ||\mathbf{x}||^2 + ||i\mathbf{x}||^2$, even though $< \mathbf{x}, i\mathbf{x} >\neq 0$. (c) $< \mathbf{x}, \mathbf{y} >= 0$ whenever $||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$ and $||\mathbf{x} + i\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||i\mathbf{y}||^2$.
(5) Using Cauchy-Schwarz Inequality for appropriate vector space with suitable inner product etc. derive the following versions of CS-inequalities:
(a) $|\Sigma x_k \bar{y_k}| \leq (\Sigma |x_k|^2)^{\frac{1}{2}} (\Sigma |y_k|^2)^{\frac{1}{2}}$. (b) $| \int_0^1 f(x)\overline{g(x)}dx| \leq (\int_0^1 |f(x)|^2 dx)^{\frac{1}{2}} (\int_0^1 |g(x)|^2 dx)^{\frac{1}{2}}$ (c) $|tr(AB^\star)| \leq (tr(AA^\star))^{\frac{1}{2}} (tr(BB^\star))^{\frac{1}{2}}$.
(6) Let $A, B \in M_{n\times n}(\mathbb{R})$ be two square matrices of size $n$ over reals. Verify that an inner product on $M$ is $< A, B >= tr(AB^t)$. Determine the subspace $W^\perp$ for $W = \{A \in M_{n\times n}(\mathbb{R}) \mid A^t = A\}$.
(7) Show that $< A, B >= tr(AB^\star)$ defines an inner product on $M_{n\times n}(\mathbb{C})$. Determine $W^\perp$ for $W = \{A \in M_{n\times n}(\mathbb{C}) \mid A^\star = A\}$.
(8) Show that $< f(x), g(x) >= \int_{-\pi}^{\pi} f(x)g(x)dx$ defines an inner product in $C[-\pi, \pi]$. Define $\mathbf{1}(x) = 1, \forall x \in [-\pi, \pi]$. Show that

$$S = \{\mathbf{1}\} \cup \{\cos(mx) : m \geq 1\} \cup \{\sin(nx) : n \geq 1\}$$

is a linearly independent subset of $C[-\pi, \pi]$.
(9) Determine an orthonormal basis of $\mathbb{R}^4$ containing the vectors $(1, -2, 1, 3)$ and $(2, 1, -3, 1)$.
(10) Consider the real inner product space $C([-1, 1])$ with $< f, g >= \int_{-\pi}^{\pi} f(t)g(t)dt$. Show that the polynomials $1, x, \frac{3}{2}x^2 - \frac{1}{2}, \frac{5}{2}x^3 - \frac{3}{2}x$ form an orthogonal set in $C([-1, 1])$. Find the corresponding functions $f(x)$ with $||f(x)|| = 1$.