SC116- Algebraic Structures

Home Work 3

Week: August 5, 2013

Tutorial Discussion Week: August 12, 2013

(1) For the matrix A, shown below, find its row space $\mathcal{R}(A)$ and its column space $\mathbb{C}(A)$ and hence find row-rank and column-rank of A. Are they same?

$$A = \begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix}.$$

(2) Let $V = \{(x, y) : x, y \in \mathbb{R}^2\}$. For $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in V$, define

$$\mathbf{x} + \mathbf{y} = (x_1 + x_2, y_1 + y_2) \text{ and } \alpha\mathbf{x} = (\alpha x_1, 0) \text{ for all } \alpha \in \mathbb{R}.$$

Is V a vector space? Give reasons for your answers.

(3) Determine all the subspaces of $\mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}^3$.

(4) Which of the following are correct statements (why!)?

(a) $S = \{(x, y, z) \in \mathbb{R}^3 : z = x^2\}$ is a subspace of $\mathbb{R}^3$. (b) $S = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1$ is rational$\}$ is a subspace of $\mathbb{R}^n(\mathbb{R})$. (c) $S = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : |x_1| \leq 1\}$ is a subspace of $\mathbb{R}^n(\mathbb{R})$. (d) $S = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1| = |z_2|\}$ is a subspace of $\mathbb{C}^n(\mathbb{R})$. (e) $S = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : z_1$ is real$\}$ is a subspace of $\mathbb{C}^n(\mathbb{C})$.

(5) Let $S = \{(1, 1, 1, 1), (1, -1, 1, 2), (1, 1, -1, 1)\} \subset \mathbb{R}^4$. Does $(1, 1, 2, 1) \in L(S)$? Furthermore, determine conditions on x, y, z and w such that $(x, y, z, w) \in L(S)$.

(6) Find 3 vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^4$ such that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent whereas $\{\mathbf{u}, \mathbf{v}\}, \{\mathbf{u}, \mathbf{w}\}$ and $\{\mathbf{v}, \mathbf{w}\}$ are linearly independent.

(7) Show that any set of k vectors in $\mathbb{R}^3$ is linearly dependent if $k \geq 4$.

(8) Is $\{(1, 0), (i, 0)\}$ a linearly independent subset of $\mathbb{C}^2(\mathbb{R})$?

(9) Let $P_n(\mathbb{R}) = \{$Set of all polynomials of degree at most n with real coefficients$\}$ and let $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{i=0}^{n} b_i x^i \in P_n(\mathbb{R})$, for some $a_i, b_i \in \mathbb{R}, 0 \leq i \leq n$. Define vector addition and scalar multiplication as

$$f(x) + g(x) = \sum_{i=0}^{n} (a_i + b_i)x^i \text{ and } \alpha f(x) = \sum_{i=0}^{n} (\alpha a_i)x^i, \text{ for } \alpha \in \mathbb{R}.$$

Determine a basis of $P_n(\mathbb{R})$ and hence find its dimension.

(10) Consider the vector space $C([-\pi, \pi])$. For each integer n, define $\mathbf{e}_n(x) = \sin(nx)$. Prove that $\{\mathbf{e}_n : n = 1, 2, 3, \ldots\}$ is a linearly independent set.