Mathematics Teaching Beyond 2000: Some Loud Thinking*

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The Problems

Looking back on the last century, one can safely say, as one could have said at the turn of the earlier one, *and* the one previous to that, that something big has been gathering on the horizon. Flashes have been seen but the rumblings have yet to roll down. They will arrive for sure. The rumblings of the last two centuries keep circulating undiminished, adding "timbre" to our thoughts, and they have already begun to have those of the present one join in for accompaniment. We will do well not to let pedantry come, even inadvertently, in the way of the young getting to "hear the new sounds". It is a difficult task, but a task we cannot shirk. Indeed, to decide amongst the myriad notes old and new, which are the key notes, and which ones are subsidiary, will call for an intense and sustained collective effort. To include or not to include would be the question.

There would appear to be three broad fronts on which we need to work. These fronts may be characterized thus:

- **Separated by a Common Purpose:** Amidst unity on the need to teach mathematics as part of the common core, there is diversity of views, mutually conflicting at places and inevitably so, on what all should in the current era be taught, what should go and what should come in its place. Maybe the different things that people want are not that different after all. There is a need to thrash out basic issues of this kind through workshops and seminars, with all disciplines participating.

- **Problems of Scale:** Large classes, large tutorial sections, tutors drawn from various different departments and disciplines, large gaps between skills and knowledge–base that core tutorials demand of the tutors and those that they commonly use in their own professional sphere for teaching or research, widely differing views on the magnitude and significance of the work load demanded by assignment problems — all these are problems of scale that require a fresh, vigorous and sustained initiative to resolve, more so now than a decade earlier.

*A partially reorganized version of notes first written in August 1999. Tex File: C:/vps_tex/BOOK/math2005.tex.

- **Understanding *vs* Application — Truths of Logic and Truths of Life:** Truth in logic is relative; it is what propagates from the constituents of a proposition to the proposition itself, or from axioms to theorems to theorems. To understand is to see how such truths propagate.

  Truths of reality, of statements about the physical world, on the other hand, lie in meanings, and in what is borne out by the goings on around us. To act and to apply our physical and mental abilities to deal with worldly issues is to correlate such truths.

  Both kinds are important in their own way. But, for truths of logic to evoke wide interest, they must be *seen* to ultimately help us comprehend the overall picture that truths of reality form. Curricula need to be so fashioned as to reflect this fact.

Looking for Answers

In order to fruitfully engage in efforts on these fronts, it would seem worth while to try to take stock of (a) the new perspectives that are discernible in mathematics *per se*, in its relationships with other disciplines and (b) of the vast array of mediatory computational tools, both for action and for communication, that the new technologies have brought in their wake.

As to new perspectives in mathematics *per se*, there are strong undercurrents of change and revision that have the potential to influence teaching of mathematics in a major way. I might just mention a few that have come to my notice.

- **Computable This and That:**

  While the stirring contributions of Brouwer, Gödel, Turing, Herbrand, Church, and other luminaries are, in their full glory, a matter for the specialists to ponder, there are pragmatic residues of the debates they have generated that are going to influence the practice of mathematics. One of them is the notion of computable numbers, functions etc.There is already a substantial body of literature on rewriting Analysis, in which the notion of *existence* has to be laid out in terms of the notion of algorithms, and in which to define is to give an algorithm for evaluation. On this point, let me quote Aberth.[1]

[1] Oliver Aberth [6, p.2]

> "Computable analysis may be informally described as an analysis wherein a computation algorithm is required for every entity employed. The functions, the sequences, even the numbers of computable analysis are defined by algorithms. In this way definition and evaluation are inseparable.
> A real number may figure in the analysis—that is, it is a computable number—if there is an algorithm for obtaining arbitrarily precise rational approximations to it. ... Also, it is not difficult to see that sum, difference, or product of two computable numbers is again a computable number— the quotient also if the divisor is not zero. In short, the computable numbers form a field.
> It may appear that all real numbers must be computable numbers, but a simple counting argument of the kind often made in real analysis disproves this. In an appropriate finite alphabet an algorithm may be specified by a finite number of symbols. Thus the number of algorithms is not larger than the number of finite messages that can be written using the alphabet. But the number of such messages is obviously denumerable, and so there can be no more than a denumerable number of computable numbers."

The question that arises is whether we are going to have in our mathematics teaching some scope for admitting such ideas, and if yes, then how. Do these in any way call for a reorientation even at the undergraduate level?

- **Enter the Infinitesimal:** For numbers as we commonly know and teach them, there is no such thing as being small or large — the same number may be small in one context and large in another. We can talk of one being smaller than another, but not of one being small on its own.

  But we do all the same talk of them in practice in this way; what we usually mean is that a particular number is, in comparison to all possible numbers that are likely to arise in the context of a particular computation, small enough to be negligible, and there is therefore no harm in that computation to neglect it.

  Well then, why not idealize a bit and think of numbers, *the infinitesimals*, that are not zero but are smaller than any positive (real) number that might arise in any computation whatever? Agreed that there is no real number to fit the bill, but that only means we need to introduce another number system in the hierarchy of number systems that we already have; if it helps, that is. The passage from naturals to integers was unnatural at one time, but it helped. And so was the passage to reals and complex, but it helped.

  We are assured that we are on safe grounds in doing so, and that the small quantities of the practitioner are formally justifiable after all. The original line of justification was too esoteric, enmeshed in logic and model theory as it was, but a good deal of polishing has led to other versions that appear to be more accessible.[2] The upshot of all this is that there seems to be in the offing a rendering of calculus in a language that may turn out to be intuitively more appealing (more practice–friendly, as it were) than the classical ϵ–δ language.

Do we see the need for any preparatory shifts in our treatment of calculus to accommodate the resulting changes when they come?

- **Sets Minus the Axiom of Well–Foundedness:** Then there are voices on the fringe telling us that it is a good thing to let sets be members of themselves. Calling them hypersets when they are so,[3] they assure us that in modelling circular phenomena, we are better off with them as part of our inventory. Practical situations arising in the study of relational databases, relations in general, and graphs are placed as evidence.

  Does this mean that discrete mathematics has far deeper significance for classical mathematics than what is commonly thought?

As to computational tools, there is first the notion of computation itself. Computation at one time meant straight forward numerical work — usual arithmetic, numerical matrix algebra, numerical solution of equations of different kinds. On the margin, it also meant string manipulation, as in record keeping and database management.

Today, it is all that and much more. Besides many other things,[4] it is also symbolic handling of algebra and Analysis. There are various kinds of software tools available for symbolic computation that enable us to do calculus and algebra as we normally do.[5] Of the underlying theory,[6] and of their applications,[7] there is a good deal new that needs to be brought in within the scope of what we teach under computation.

Incorporating such software tools, in addition to those generally used for numerical work (e.g., MATLAB, SCILAB), with easy availability and access to students, in our mathematics teaching scheme would greatly enhance our effectiveness, release our energies to concentrate more on concepts, and allow wider margins of effort to students to indulge in active learning, supplementing what they do passively through lectures and tutorials.

[2]For a full–blown account of the ideas as originally conceived and put together as non-standard analysis, there is Robinson [1]; it also has a good historical account of the calculus and the status of the infinitesimal from the times of Newton and Leibniz. Hoskins [2] presents the basic ideas in a more familiar setting. A brief an elegant article on the subject is [3].

[3]The article by Barwise and Ross [4] presents the case for such sets very lucidly.

[4]Like word–processing, desk–top publishing, computer aided design and manufacturing, computer graphics, signal representation and processing, traffic control and management, telephony and satellite communication, plant control etc., etc.

[5]For example, MAPLE, MATHEMATICA and AXIOM for symbolic computations in general, AREP and GAP for group theoretic symbolic computation; the power of AXIOM is explained in [5], but I am not sure at this time whether it is still marketed.

[6]I have in mind such developments in algebra as are discussed in [7].

[7]A good illustration of this is in the study and solutions of polynomial equations in several variables via the theory of *algebraic varieties*, as discussed for instance in [8].

In Summary

While prudence is well–advised when it comes to admitting radically new ideas and new trends into our basic mathematics curriculum, it is reasonable to believe that the new technological developments provide us very useful tools to make our teaching, even of the classical mathematical concepts, far more effective.

We need to encourage an environment in which computation (in the wider sense mentioned earlier) oriented infrastructural support is freely and abundantly available for interactive teaching and learning of mathematics.

A Collage of Ideas

Before closing, let me reproduce here some very interesting comments by some of the veterans.[8]

- **On Being Exact**:[9] Consider the simplest kind of integration problem, say $\int_0^{13} e^x dx$. It's the kind of problem you put on a freshman calculus exam so that every student should get at least one problem right. Everybody knows how to do it. The answer is $e^{13} - 1$. Suppose someone came along and said, "Why didn't you use numerical integration?" You will say that is absolutely mad. Here is a closed form solution, and you can get the exact solution in two minutes; in 10 seconds if you are good. Why in heaven's name would you want to use numerical integration to get an approximate answer? So my question is, if that's the exact answer, please tell me what it is to one significant figure. (I am sure there are people in the room, who, if they thought about it long enough, could get it, but, as a coward, I am afraid I went to a computer.) The answer to one significant figure is 400,000.

  Next you will argue that it is still important that $e^{13} - 1$ is exact. You know, for years I accepted that, until one day I woke up, and it hit me that the original integral $\int_0^{13} e^x dx$ is also exact! I do not mean that as a joke – I mean that as a deep remark about mathematics, one that I overlooked for a large number of years. Now the question is, *if you have two forms that are exact, why is one preferable to the other one* (my italics)?

- **Symbolically Yours:**[10] Because of computers, something old can go and something new is coming. That much is certain. The disagreements begin when we start making lists, but in the interest of provoking debate, I will take the plunge. ...First we'll discuss the impact on the curriculum of the computers that do symbolic manipulation. ...(And then) the teaching of algorithms in the early years of college.

[8] The captions are mine.

[9] John J. Kemeny, "Finite mathematics – then and now" in [9, pp. 201–208]

[10] Herbert S. Wilf, "Symbolic manipulation and algorithms in the curriculum of the first two years" in [9, pp.27–42].

. . . First, some portions of the curriculum presently teach concepts or methods that computers will soon be able to handle even better than people can. Of course, just because a computer can do it doesn't imply that people shouldn't learn it. But *some* things are of that type: not only computers do them better, but they have little or no redeeming intellectual value, so why not relax and let the machines carry the ball?

Secondly, the pressure of the ideas that must enter the curriculum is, in certain cases, so intense that some topics will have to be deleted just to make time available for the persistent newcomers. In other words, even though the dropped portions may retain a good deal of value, they won't be able to compete with the thrust of some of the new ideas that are screaming for attention.

Let us begin with some areas of least disagreement, hopefully. . . .

. . . In the first year we probably could cut the total time spent on methods of integration by half, so that each method gets just a quick once–over–lightly, two problems to do without calculators, and a few to do with calculators.[11] With some others small economies of this kind, having to do with drilling skills, I would estimate that the first year course could shrink by 15%, *losing only the bath water, and keeping the baby*. The second year could be cut perhaps by 1/3, reflecting the usefulness of the calculators in doing the (symbolic algebraic)) manipulation parts of the work in differential equations and linear algebra. . . . To help fix ideas I will now suggest a syllabus . . .

- **Beyond Definitions:**[12] "I can't define it but I know when I see it." If Potter Stewart hadn't coined this phrase, mathematicians would have. Trying to define "mathematical maturity" is about as hopeless a task as defining obscenity, or, for that matter, justice or love. Nevertheless, since it is our task, let us begin.

  . . . There are several marks of maturity that most mathematicians will instantly recognize. One of the most important is the ability to abstract, to glean the essential structure from a complex situation. . . .

  A second mark of maturity is the ability to synthesize, to create new ideas by effective use of old ones. Logical deduction, indirect reasoning, the ability to create and follow an extended argument are key ingredients in synthesis. But so too are the elusive elements of imagination and creativity. . . .

  Teachers are eternal optimists, constantly looking for signs of spring in the bleak landscape of winter classrooms. A good question from the back of the room is like a crocus poking its head above melting snow. It is a sign of emerging maturity. Here are some other signs:

  * The ability to use and interpret mathematical notation.

[11]Writing in 1983, when the power that we have today of PC's, networked computers and the associated application software was still to come, he places himself in the future, imagines that a portable little machine with all that power is available, and calls it a calculator.

[12]Lynn Arthur Steen, "Developing mathematical maturity" in [9, pp.99–110]

* The ability to model, to express real–world problems in mathematical form.
* The ability to perceive patterns, and to apply principles of symmetry.
* The ability to estimate, to solve problems by perturbing the data.
* The ability to generalize, to infer fundamental laws from particular cases.
* The ability to detect and avoid sloppy reasoning.
* The ability to see and exploit relationships among various parts of mathematics.
* The ability to read and understand mathematical writing , whether in mathematics or in scientific contexts.

- **Axioms Unspoken:**[13] Cartwright suspects that Littlewood may have had some unspoken rules for their collaboration comparable to those he had with Hardy. Harold Bohr noted that the following four "axioms of collaboration" formed a basis for the Hardy–Littlewood relationship.

  1. When one wrote to the other, it was completely indifferent whether what they wrote was right or wrong.
  2. When one received a letter from the other, he was under no obligation whatsoever to read it, let alone answer it.
  3. Although it did not really matter if they both simultaneously thought about the same detail, still, it was preferable that they should not do so.
  4. It was quite indifferent if one of them had not contributed the least bit to the contents of the paper under their common name.

  ... Littlewood often thought in dynamical terms, perhaps because of his earlier work in ballistics during the First World War. She (Cartwright) says that he often referred to solutions as "trajectories" as if they were paths of missiles fired from antiaircraft guns. He certainly seemed to understand the dynamical aspects of the radio work. Van der Pol, recalling discussion with Littlewood, told Cartwright in "tones of delighted surprise" that all of Littlewood's methods corresponded to physical concepts.

- **Recognition:**[14] The development of science is not a simple progression from one advance to the next. Judged by hindsight, the development is slow, proceeds in a zigzag course, with many wrong turns and blind alleys, and frequently moves in directions condemned by leading scientists. In the 1930's Banach spaces were sneered at as absurdly abstract, later it was the turn of locally convex spaces, and now it is the turn of nonstandard analysis.

- **Rigour:**[15] In order to formulate a more careful argument, I need to say

[13] From [10]
[14] From [11].
[15] Excerpted from [12].

a few words about the concept of *rigour*. It is widely believed that this notion changes. Arguments that seemed rigorous to Euler seemed inadequate to Cauchy. Arguments that seemed rigorous to Cauchy seem to us to contain obvious gaps. But it is not really the case that the concept of rigour has changed—only the standard of rigour. That is to say, a rigorous argument is always an argument which suffices to establish the truth of its conclusion. As our insight grows, we see that more is required to establish truth, and therefore arguments that once seemed rigorous are now seen to have gaps. But the concept of rigour itself has not changed since at least the time of Euclid.

More is true than that the concept of rigour presupposes the concept of truth. Actually, when we evaluate a mathematical argument, we do not check to see whether it accords with some set of rules taken, let us say, from a logic text. Rather, we try to determine whether the argument works—that is, whether it convinces us, and it ought to convince us, of the truth of its conclusion. Thus the concept of mathematical truth is directly involved in the practice of mathematical rigour. It functions as an indispensable ingredient in the very criterion of rigour. . . . (It) is practically real (in the sense that he says earlier about an attribute, "I mean that the attribute is practically real if it plays a role, and that there exists a consensus that the attribute does play a role and should play a role, in our interactions with the objects that have the attribute."). Indeed, without the practical reality of mathematical truth, there would be no such thing as mathematical rigour.

- **What of That?:**[16] "Mathematics is an experimental science and definitions do not come first but later on. . . . First get on, in any way possible, and let logic be left for later work. . . . The use of operators frequently affects great simplifications, and the avoidance of complicated evaluations of definite integrals. But then the rigorous logic of the matter is not plain! Well, what of that? Shall I refuse my dinner because I do not fully understand the process of digestion?"

  Insistence on ultimate rigour he called a pernicious doctrine with three pernicious results (i) it enfeebles the mind (ii) it leads to omission of works whose rigorous proofs authors can not provide (iii) leads to inability to recognize the good in other men's works that are unconventional and devoid of rigorous prooofs.

- **Easy Come Easy Go:**[17] The only way to learn mathematics is to do mathematics. That tenet is the foundation of the do–it–yourself, Socratic, or the Texas method, the method in which the teacher plays the role of the omniscient but largely uncommunicative referee between the learner and the facts. . . .

  The right way to read mathematics is first to read the definitions of the concepts and the statements of the theorems, and then, putting the book aside, to try to discover the appropriate proofs. If the theorems

[16] Taken from Moore [13, p.216], writing on Heaviside's rebuttal to objections to his operational calculus and his views in general.

[17] From the Preface in Halmos [14, p.vii]

are not trivial, the attempt might fail, but it is likely to be instructive just the same. *To the passive reader a routine computation and a miracle of ingenuity come with equal ease, and later, when he must depend on himself, he will find that they went as easily as they came* (my italics). The active reader, who has found out what does not work, is in a much better position to understand the reason for the success of the author's method, and, later, to find answers that are not in the books.

- **Catch–22:**[18] There were a handful of experiences that led me to choose mathematics as a career. One was a high school teacher showing me the continued fraction

$$x = 1 + \frac{1}{1 + \frac{1}{1+\cdots}}$$

...On the one hand was the ellipsis "$\cdots$" descending like Lucifer's band ever deeper into the nefarious denominator. On the other was the light and clarity of the solution obtained by observing that the fraction satisfies the identity $x = 1 + 1/x$. I was hooked.

...I had a similar experience a few years ago, in reading a manuscript by Peter Aczel on non–well–founded sets (hypersets).[19] ...A simple example of a hyperset is: $x = \{1, \{1, \{1, \{1, \ldots\}\}\}\}$. This set is a member of itself, for the same reason that the continued fraction satisfies the identity $x = 1 + 1/x$. Therefore, x is a hyperset. ...Sure, you can play the same game by noting that $x = \{1, x\}$. But then what? ...

...Over the past hundred years, a wealth of techniques for modeling various sorts of phenomena have been developed within set theory. ...However, when we try to apply these techniques to phenomena that involve circularity, the Axiom of Foundation gets in the way. ...

So the theory of hypersets must be based on new axioms. How are new axioms of set theory born? For us, axioms are conceived in an attempt to capture the essence of some important phenomenon. We want to abstract from a repeated pattern in the physical world or the world of mathematical ideas and say something true and fundamental. It is not enough that a new axiom not lead to a contradiction (as sometimes happens in set theory). Even if it were "obviously" inoculated against the plague of paradox, a new axiom will not survive unless it is based on solid intuitive understanding of *something*. Otherwise it would die stillborn for lack of interest. And indeed, the literature is strewn with stillborn theories without an intuitive conception on which to rest.

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[18] From Barwise [4]

[19] As defined in the text, a set b is a *hyperset* if there exists an infinite descending sequence, $b \ni a_1 \ni \ldots \ni a_n \ni a_{n+1} \ni \cdots$.

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