CT321: DSP
Lecture 1 & 2: Signals
AT

If an engineer is asked What is a signal?, you will usually get answers like, signal is a function of time, function of n variables, a real function, a sequence of real numbers etc. Well, all of these are not really signals, they are different ways of representing a signal. Signal is any physical (real-world) entity that contains information. In order to model the processing of signals as a black box, we need to choose an appropriate mathematical structure for modeling the signals such that it captures all the *relevant* relations between signals under consideration.

As an example let us consider speech signals. The real world speech signals can be made louder or can be muted. Similarly if two people speak together then we obtain a mixed speech signal. We now map this set of signals to a particular mathematical entity, let us say to a set of functions of one variable with some operations defined on this set. This is done via a transducer, in this case a microphone. It converts pressure on its sensor to equivalent voltage. Now we need to look for a justification in the choice of the mathematical entity. Is there an operation on this mathematical entity that corresponds to change of volume? What about mixing of speech signals. As we all know, answers to both questions is a 'yes'. Multiplying a real number to the function, and point-wise addition of two functions is what we are looking for.

On the other hand one may say, there is much more that can be done on a set of functions, for example point-wise multiplication of two functions. Therefore defining the mathematical model of our choice also includes defining what operations we intend to carry out on the set along-with the properties of the defined operations. This $3-$tuple, a set, operations and its properties define what we call a *mathematical structure*. For example, if the signal of our interest is weight on a balance then we may choose $(\mathbb{R}, +)$, where $+$ corresponds to the real world act of putting two things together. We know that we can also multiply two real numbers, but the operation is irrelevant to the signals we are concerned here.

In this course we are going to mainly focus of signals that can be modeled by some space of functions of one variable. There is still a big choice to make. What is the class of functions under consideration? All of us have come across some of them: periodic functions, functions with finite energy, functions that have a Fourier representation, functions whose Fourier transform has a finite support etc. The domain and range of the function can also be different. Sampled signals are functions from the set of integers to the set of real numbers. Digital signals are functions from the set of integers to the set of integers(henceforth *sequences*). Anybody with little background in algebraic structures will see that these are vector spaces, and all of DSP will deal with only the vector space structure.

That does not mean other structures have no relevance in DSP. For example, look at the image in figure 1 What you see over here is a slice of somebody's brain. On each point on the image there is an ellipsoid representing a 3×3 covariance matrix of water diffusion modeled as a Gaussian. The length of the axes gives the distance covered by a water molecule in some fixed time interval Δ through diffusion starting at the center point, in the direction given by the corresponding axes. Interpretations for different ellipsoids are given in the figure 2 It is well known that a covariance matrix is symmetric positive semi-definite matrix and equally well known is the fact that the set of symmetric positive semi-definite matrices forms a convex subset of $\mathbb{R}^{3\times3}$. What prevents this subset from being a linear

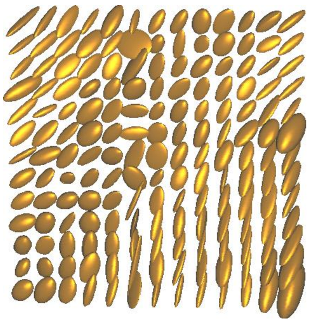

Figure 1: Diffusion Tensor Image, from `http://www.mathworks.com/matlabcentral/fileexchange/27462-diffusion-tensor-field-dti-visualization`

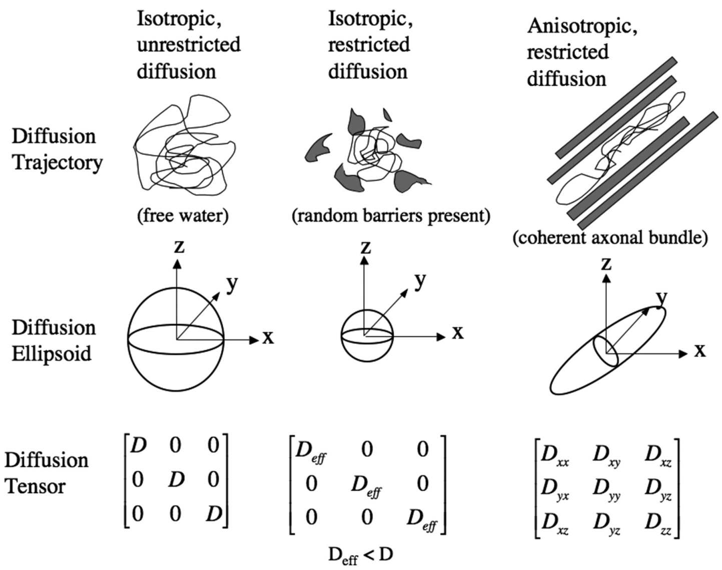

Figure 2: Ellipsoids for DTI images, from `http://www.ajnr.org/content/29/4/632/F4.expansion.html`

subspace is that it is not closed to scalar multiplication, i.e., multiplying a positive semi definite matrix with a negative real number gives a negative semi-definite matrix. Therefore we cannot use concepts from linear systems to work with such signals. Another such example is set of signals that can be represented as unit vector fields on a subset of $\mathbb{R}^2$. we know that addition of unit vectors will not yield unit vectors. Here again we will need to use a different algebraic structure.

This was just to make you aware that vector spaces though popular, are not the *universal* choice of algebraic structure for signal processing. I do not expect you to understand the details of the above examples, it is just an effort to familiarize you to practical examples involving various algebraic structures. Having said that, for the purpose of this course, we are going to work mainly with vector spaces.

Sampling

Sampling can be thought of as a map from space of functions with domain and range being the set of real numbers to the space of functions with domain and range being the set of integers and set of real numbers respectively. In usual discussions on sampling, the space of functions considered is the functions with finite bandwidth. The mapping to the space of sequences is then invertible if the sampling frequency satisfies the famous *Nyquist criteria*. If not, then the inverse map will give a function that will not be the original function. Of course, a human being might not be able to *sense* any difference and therefore aliasing is more a mathematical concept.

If we choose a different function space to model the signal of interest, fewer samples might be enough. This is what typically happens in a Compressive sensing framework. The functions under consideration are those which could be written as a linear combination of a sparse subset of basis functions. Figure 3 illustrates Compressive Sensing. Hence, it becomes all the more important to

Figure 3: (left) Original Image $128 \times 128 = 16384$ pixels (center) Reconstructed image from 1600 measurements (right) Reconstructed image from 3300 measurements. Images from `http://dsp.rice.edu/cscamera`

look at signal processing from an abstract point of view.