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THE ANT ON A TAUT RUBBER ROPE PROBLEM MODEL

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Abstract

In this article, I shall discuss the analytical way of solving the classical 'ANT ON A TAUT RUBBER ROPE PROBLEM'. This document itself does not go into much depth, but is instead the output of a model to demonstrate the structure of the universe and the principles of space-time.

1 INTRODUCTION

‘‘ Ant on a rubber rope ” is a mathematical puzzle with a solution that appears counter-intuitive or paradoxical. It is sometimes given as a worm, or inch-worm, on a rubber or elastic band, but the principles of the puzzle remain the same.

The details of the puzzle can vary, but a typical form is as follows. [1] [2] [3]

“ An ant starts to crawl along a taut rubber rope 1 km long at a speed of 1 cm per second (relative to the rubber it is crawling on)created by. At the same time, the rope starts to stretch by 1 km per second (so that after 1 second it is 2 km long, after 2 seconds it is 3 km long, etc). Will the ant ever reach the end of the rope ? ”

At first consideration it seems that the ant will never reach the end of the rope, but in fact it does (although in the form stated above the time taken is colossal). In fact, whatever the length of the rope and the relative speeds of the ant and the stretching, providing the ant's speed and the stretching remain steady the ant will always be able to reach the end given sufficient time.

2 A FORMAL STATEMENT OF THE PROBLEM

The problem as stated above requires some assumptions to be made. The following fuller statement of the problem attempts to make most of those assumptions explicit.

Consider a thin and infinitely stretchable rubber rope held taut along an x-axis with a starting-point marked at $x = 0$ and a target-point marked at $x = c$, $c > 0$.

At time $t = 0$ the rope starts to stretch uniformly and smoothly in such a way that the starting-point remains stationary at $x = 0$ while the target-point moves away from the starting-point with constant speed $v > 0$.

A small ant leaves the starting-point at time $t = 0$ and walks steadily and smoothly along the rope towards the target-point at a constant speed $\alpha > 0$ relative to the point on the rope where the ant is at each moment.

Will the ant reach the target-point?

3 AN ANALYTICAL SOLUTION

A key observation is that the speed of the ant at a given time $t > 0$ is its speed relative to the rope, i.e. α , plus the speed of the rope at the point where the ant is. The target-point moves with speed v, so at timecreated by t, it is at $x = c + vt$. Other points along the rope move with proportional speed, so at time t the point on the rope at $x = X$ is moving with speed $\frac{vX}{c+vt}$. So if we write the position of the ant at time t as $y(t)$, and the speed of the ant at time t as $y'(t)$, we can write:

$$\mathbf{y'(t)} = \alpha + \frac{\mathbf{v\,y(t)}}{\mathbf{c+vt}}$$

This is a first order linear differential equation, and it can be solved with standard methods. However, to do so requires some moderately advanced calculus. A much simpler approach considers the ant's position as a proportion of the distance from the starting-point to the target-point.[2]

Consider coordinates ψ measured along the rope with the starting-point at $\psi = 0$ and the target-point at $\psi = 1$. In these coordinates, all points on the rope remain at a fixed position (in terms of ψ) as the rope stretches. At time $t \geq 0$, a point at $x = X$ is at $\psi = \frac{\mathbf{X}}{\mathbf{c+vt}}$, and a speed A, relative to the rope in terms of x, is equivalent to a speed $\frac{A}{c+vt}$, in terms of ψ. So if we write the position of the ant in terms of ψ at time t as $\phi(t)$, and the speed of the ant in terms of ψ at time t as $\phi'(t)$, we can write:

$$\phi'(\mathbf{t}) = \frac{\alpha}{\mathbf{c+vt}}$$

$$\therefore \phi(\mathbf{t}) = \int \frac{\alpha}{\mathbf{c+vt}}\,\mathbf{dt} = \frac{\alpha}{\mathbf{v}}\log(\mathbf{c+vt}) + \kappa$$

where κ is a constant of integration.

Now, $\phi(0) = 0$ which gives $\kappa = -\frac{\alpha}{v}\log c$, so

$$\phi(\mathbf{t}) = \frac{\alpha}{\mathbf{v}}\log\left(\frac{\mathbf{c+vt}}{\mathbf{c}}\right).$$

If the ant reaches the target-point (which is at $\psi = 1$) at time $t = T$, we must have $\phi(T) = 1$ which gives us:

$$\frac{\alpha}{\mathbf{v}} \log\left(\frac{\mathbf{c} + \mathbf{vT}}{\mathbf{c}}\right) = \mathbf{1}$$

$$\therefore \mathbf{T} = \frac{\mathbf{c}}{\mathbf{v}}\left(\mathbf{e}^{\mathbf{v}/\alpha} - \mathbf{1}\right)$$

As this gives a finite value T for all finite $c, v, \alpha (v > 0, \alpha > 0)$, this means that, given sufficient time, the ant will complete the journey to the target-point. This formula can be used to find out how much time is required.
For the problem as originally stated, $c = 1\,\mathrm{km}$, $v = 1\,\mathrm{km/s}$ and $\alpha = 1\,\mathrm{cm/s}$, which gives

$$\mathbf{T} = (\mathbf{e}^{\mathbf{100{,}000}} - \mathbf{1})\,\mathrm{s} \approx \mathbf{2.8} \times \mathbf{10}^{\mathbf{43{,}429}}\,\mathrm{s}$$

This is a truly vast timespan, vast even in comparison to the estimated age of the created byuniverse, and the length of the rope after such a time is similarly huge, so it is only in a mathematical sense that the ant can ever reach the end of this particular rope.

$\therefore$ THE PROBLEM IS SOLVED.

4 APPLICATIONS OF THE PROBLEM

This problem has a bearing on the question of whether light from distant galaxies can ever reach us if the universe is expanding.

If the universe is expanding uniformly, this means that galaxies that are far enough away from us will have an apparent relative motion greater than the speed of light.[3] This does not violate the relativistic constraint of not travelling faster than the speed of light, because the galaxy is not ”travelling” as such – it is the space between us and the galaxy which is expanding and making new distance. The question is whether light leaving such a distant galaxy can ever reach us, given that the galaxy appears to be receding at a speed greater than the speed of light.

By thinking of light photons as ants crawling along the rubber rope of space between the galaxy and us, it can be seen that just as the ant will eventually reach the end of the rope, given sufficient time, so the lightted by from the distant galaxy will eventually reach earth, given sufficient time.

5 CONCLUSION

To summarise, the answer to the original question as stated at the beginning of the project is: yes, the ant will always reach the end of the string provided that the string is finite and is expanding at a finite rate and the ant has finite speed. Similarly, light from the distant galaxies will always eventually reach earth , given sufficient time.

References

[1] Martin Gardner. aha! Gotcha: paradoxes to puzzle and delight. W. H. Freeman and Company, 1982.

[2] The Problem Site. The Long Walk. Maths puzzles on the web, http://theproblemsite.com/problems/mathhs/2002/Oct_1.asp, 1-10-2002.

[3] William Wu. Walking on a Stretching Rubber Band. Wu:riddles forum, http://www.ocf.berkeley.edu/~wwu/riddles/hard.shtml, 17-1-2003.